1950

Notes on continuous beam report, September 1950

W. H. Weiskopf

(1) Harry
(2) LSB (205 B Report)

WALTER H. WEISKOPF
M. AM. SOC. C. E.

HENRY HOFELDT
M. AM. SOC. C. E.

WEISKOPF & PICKWORTH
CONSULTING ENGINEERS
45 WEST 45TH STREET
NEW YORK 19, N. Y.

~~JOHN W. PICKWORTH~~
M. AM. SOC. C. E.

The report is old and diss. is more complete and better

September 19, 1950

Mr. Lynn S. Beedle
Fritz Engineering Laboratory
Lehigh University
Bethlehem, Pa.

Re: Continuous Beam Report

Dear Mr. Beedle:

I have read with great interest the draft of Section 4d of your report. I realize it is only a rough draft, but generally it covers the matter well.

I have a few comments marked in red on the draft and would like to mention a few more as follows:

(1) Page 3, first paragraph "...... the moment at A and C equals 1500 kips. This value is very close to the plastic hinge moment 1497 in-kips." This at first glance gives the impression that the simple plastic theory is extremely accurate, but this is not the case since the values compared are for materially different loads, 41 kips and 50.5 kips. The comparison seems to me quite meaningless and I suggest it be omitted.

(2) Page 7, last paragraph "Methods of deformation and strength for structures in plastic range discussed above can be summarized as follows:" Method b has not been "discussed above". It seems to me a better arrangement would be to give method b earlier and in the summary simply refer to it.

(3) The approximation suggested by you (method d) of eliminating the plastic range, I think is very good. I tried this out last fall and was considering suggesting it. The enclosed prints, dated 11-6-49, show a comparison of this method on Test specimen B2. On sheet 3 you will see that the agreement is very good. However, a word of warning. The approximation works well where the moment diagram is steep and therefore the plastic range is small. But if the moment diagram is flat or horizontal where the moment is maximum, the plastic range will extend over a long length of beam and the approximate method may then be inaccurate. Such a case for instance is third point loading where the end restraint is such that the maximum moment is in the middle of the beam as in Test B3.

The appendix is being prepared and will be sent to you in a few days.

Regards to Mr. Yang and yourself.

Yours truly

cc: Mr. T. R. Higgins

WHW:F

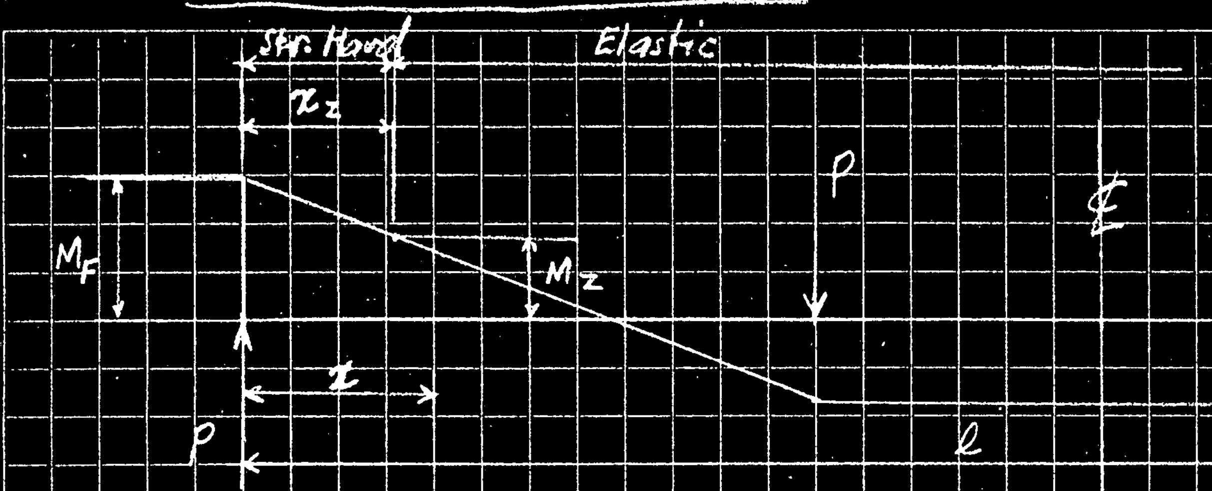
Job No.

Designed by W

Sheet No. 1 of 3

Date 11-6-49

APPROX METHOD



For elastic section

$$\frac{dy}{dx} = \frac{1}{EI}\left[M_F\left(x - \frac{l}{2}\right) + P\left(\frac{l^2}{9} - \frac{x^2}{2}\right)\right]$$

$$x_z = \frac{M_F - M_z}{P}$$

For str-hard section

$$\frac{dy}{dx} = \frac{1}{BI}\left[M_F x - \frac{Px^2}{2} - AZx\right]$$

When $x = x_z$ sections meet

$$\frac{1}{EI}\left[M_F\left(x_z - \frac{l}{2}\right) + P\left(\frac{l^2}{9} - \frac{x_z^2}{2}\right)\right] = \frac{1}{BI}\left[M_F x_z - \frac{Px_z^2}{2} - AZx_z\right]$$

$$M_F\left[\left(\frac{1}{E} - \frac{1}{B}\right)x_z - \frac{l}{2E}\right] + \frac{Pl^2}{9E} - P\left[\left(\frac{1}{E} - \frac{1}{B}\right)\frac{x_z^2}{2}\right] + \frac{AZ\,x_z}{B} = 0$$

$$-\left(\frac{1}{B} - \frac{1}{E}\right)\frac{M_F^2 - M_F M_z}{P} - \frac{M_F l}{2E} + P\left(\frac{1}{B} - \frac{1}{E}\right)\frac{M_F^2 - 2M_F M_z + M_z^2}{2P^2} + \frac{AZ(M_F - M_z)}{B\,P} + \frac{Pl^2}{9E} = 0$$

$$-\frac{M_F^2}{2P}\left(\frac{1}{B} - \frac{1}{E}\right) + M_F\left(-\frac{l}{2E} + \frac{AZ}{BP}\right) + \left(\frac{1}{B} - \frac{1}{E}\right)\frac{M_z^2}{2P} - \frac{M_z AZ}{BP} + \frac{Pl^2}{9E} = 0$$

$$-\frac{M_F^2}{2}\left(1 - \frac{B}{E}\right) + M_F\left(-\frac{Pl}{2}\frac{B}{E} + AZ\right) + \left(1 - \frac{B}{E}\right)\frac{M_z^2}{2} - M_z AZ + \frac{P^2 l^2}{9}\frac{B}{E} = 0$$

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$$M_F = \frac{+AZ - \frac{Pl}{2}\cdot\frac{B}{E} \pm \sqrt{\left(-AZ + \frac{Pl}{2}\cdot\frac{B}{E}\right)^2 + 2\left(1-\frac{B}{E}\right)\left[\left(1-\frac{B}{E}\right)\frac{M_Z^2}{2} - M_Z AZ + \frac{P^2 l^2}{9}\cdot\frac{B}{E}\right]}}{+\left(1-\frac{B}{E}\right)}$$

$$\frac{B}{E} = \frac{667}{30{,}000} = .0222 \qquad \left(1-\frac{B}{E}\right) = .978$$

$AZ = 28 \times 39.8 = 1112$

$M_Z = 1500$

$l = 168''$

TEST B2

At beginning of str-hard range

$$P = 38.4 \times \frac{1500}{1433} = 40.1^{k.} \qquad M_Z = 1500 = M_F$$

$P = 45^{k.}$

1029 83.2

$$M_F = \frac{1112 - \frac{45 \times 168}{2} \times .022 \pm \sqrt{(1029)^2 + 2 \times .978\left[.978 \times \frac{1500^2}{2} - 1500 \times 1112 + \frac{45^2 \times 168^2}{9} \times .022\right]}}{.978}$$

$$M_F = \frac{1029 \pm \sqrt{1{,}059{,}000 + 1.956\left[1{,}100{,}000 - 1{,}668{,}000 + 140{,}000\right]}}{.978}$$

$$M_F = \frac{1029 \pm \sqrt{1{,}059{,}000 - 837{,}000}}{.978} = \frac{1029 \pm 471}{.978} = 1533''^{k.}$$

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Job No. ....
Designed by ....
Sheet No. 3 of 3
Date ....

$P = 50$

$$M_F = \frac{1112 - \frac{50 \times 118}{2} \times .022 + \sqrt{(1020)^2 + 2 \times .978[1.100.000 - 1.168000 + 173000]}}{.978}$$

$$M_F = \frac{1020 + \sqrt{1,040,000 - 772\,000}}{.978} = \frac{1020 + 518}{.978} = 1570^{k'}$$

$P = 55$

$$M_F = \frac{1112 - \frac{55 \times 118}{2} \times .022 + \sqrt{(1010)^2 + 2 \times .978[1.100.000 - 1.168.000 + 209\,000]}}{.978}$$

$$M_F = \frac{1010 + \sqrt{1020,000 - 702,000}}{.978} = \frac{1010 + 564}{.978} = 1610$$

P	M_F Approx Method	M_F Exact Method
45	1533	1525
50	1570	1565
55	1610	1606

1. Introduction

   a. Previous work

   b. Objective of this program

2. Design of the test

   a. Test set up

   b. Loading apparatus

   c. Dynamometers

   d. Lateral Support

   e. Preparation of specimen

   f. Strain gages

   g. Level & deflection gages

   h. White wash

3. Test Procedure

   a. Loading procedure

   b. Criterion used for loading in plastic range

4. Test results & Discussions

   a. Bending strength of the 8WF40 and 14WF30 sections

   b. Initial yielding strength of the beams

      Factors, residual stresses, stress concentration & welding hear effects will be briefly discussed.

   c. Ultimate strength of the beams

      Strain hardening & different restraint will be discussed.

   d. Deformation of beams in plastic range s and strain-hardening

      W.H. Weiskopf's paper will be briefly discussed.

   e. Shear failure of the beams

   f. Local buckling of compression flagnes of the beams

   g. Lateral buckling of beams

   h. Repeated loading test

5. Conclusions & summary

   a. Basic plastic behaviors of beams

   b. Plastic design theory

6. Appendix

   a. W.H. Weiskopf paper

   b.** Coupon result.

* Different welded connections.

** Yielding process of steel.

d. Deformation of Continuous Beams in Plastic Range

In elastic theory the famous Bernoulli-Euler's equation

$$\frac{d^2y}{dx^2} = \frac{1}{k} = \frac{M}{EI} \qquad \cancel{\frac{1}{EI}\frac{d^2y}{dx^2} = \phi = M}$$

has been used to find the deflection curve of flexural members for structures of small deflections. This same formula has been proposed for use in the flexural members in the plastic range provided the deformation is small. (Ref.__).

The relation between M and Ø in plastic range is of course no longer linear. But if the M-Ø ~~in plastic range is~~ relation can be determined as described in Progress Report 1 (p.__) for different types of sections, the deflection curve can be integrated numerically or use approximate analytic functions ~~to represent the~~ M-Ø curve in the plastic range.

In statically determinate structures, since the moment distribution on the member is independent of its deformation, the deflection curve of the flexural member can be easily obtained by either method.

But in statically indeterminate structures the moment distribution can not be solved unless another set of equations is added in addition to the equations of equilibrium. A set of equations considering the ~~continuity of the boundaries of the members~~ relative deflections of portions of the structure has been applied to the solution of indeterminate structures in the elastic theory. It seems ~~the same kind of~~ a similar set of equations should also be used in indeterminate structures in plastic range.

To avoid the mathematical complication, the simple plastic theory makes a further assumption for structures of structural steel. Instead of considering the ~~contuinty~~ continuity of the member at the sections that exceed the yield moment M_y it is assumed that these sections will maintain a value of plastic hinge moment, M_p, somewhat greater than M_y, in further increase of load. The section will act as a hinge. As pointed out in the previous section this will be true only if structural steel does not exhibit the strain hardening effect.

It is obvious that if this assumption is a good approximation for estimating the ultimate strength of an indeterminate structure then it should be just as good in calculating its deformation. From the test results we found this assumption is far from the actual case when the deformation of the beam becomes large.

Take the example of a built-in beam under one third point loading.

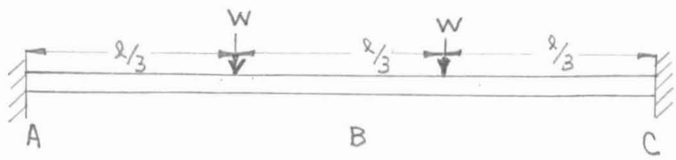

Fig a.

After w reaches its initial yield load the sections A and C start to yield. The moment at A and C ~~through~~ always increases as w increases, but the increasing rate ~~being~~ was very low, before the central part of the beam B reaches the yield point. ~~and~~ The moments at A and C stay very close to the plastic hinge moment. For the 8WF40 sections by using the simple plastic theory, put plastic hinge moment at A and C and we find the central sectinn will yield at a load of W = 50.5 kips. By using the theoretical M - Ø curve in plastic range and considering the strain-hardening effect and the continuity of the sections A and C to be fixed into the wall, the load at which the central section B reached the yield stress can be computed by numerical method. We find W = 41 kips and the moment at A and C equals to 1500 kips. This value is very close to the plastic hinge moment 1497 in-kips.

However, this assumption in simple plastic theory of neglecting the strain hardening effect is far from the truth as soon as the central portion starts to yield. In section C we have shown the tested beams had much higher end moment than the plastic hinge moment when the load was close to their collapse load, this is due to strain hardening.

Fortunately, deflections of structures usually can not afford to be too large. It is very questionable to use the collapse load as the design load. If the deflection of the structures in design had been limited within certain limits then the assumption of neglecting the strain hardening effect is expected to give a reasonable close calculated value to the deflections for the structure in plastic range.

Further simplifications in calculation will be obtained by neglecting the plastic zone near the plastic hinge. Take the built-in beam as an example.

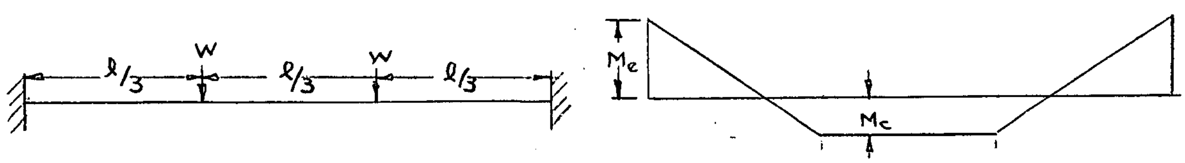

Fig b

Suppose W_1 = initial yield load

W_2 = the load when central sections start to yield

When the beam is under load W, and W is in between W_1 and W_2 (i.e., $W_1 \leq W \leq W_2$), the deflection of the beam can be found as the case of a simply supported beam with plastic hinge moments at each end.

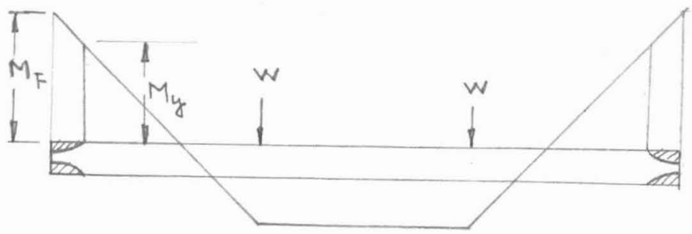

Fig c

Since $\frac{M_p}{M_y}$ = 1.1 -- 1.2 for most of the I-section the plastic zone at two ends of the beam is usually very short. The assumption of considering the above beam all elastic will not introduce too much error in computing deflections.

Fig. , the solid line is deflection of 8WF40 beam at the central section computed with the consideration of the strain hardening effect and the plastic zone at both ends by using the method of numerical integration. The points along the curve are computed according to the assumptions of simple plastic theory.

This Fig not included.

They give very close values in the range of $W_1 \leq W \leq W_2$.

Attempts have been made to solve structural problems ~~in plastic range~~ considering the strain hardening effect by use analytic function for the M-ϕ curve in plastic range for I-sections.*

?

I suggest we say "inelastic" WB 9/2?

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* Flexure of I-Sections Above Elastic Range" by Walter H. Weiskopf.

Why not say - "structural problems in the plastic and strain hardening ranges"

Weiskopf made some further assumptions* to the M-ϕ curve of I-sections and divides the M-ϕ curve to three different portions which are elastic, plastic, and strain hardening range and represents them in three different expressions.

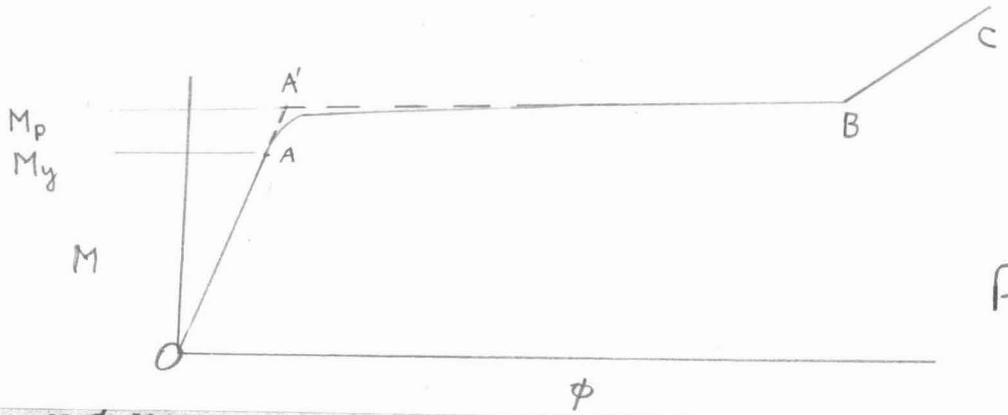

Fig d

In above M-ϕ diagram

for portion o A (straight line)

$$\phi = \frac{d^2y}{dx^2} = M/EI \qquad \text{(a)}$$

for portion A B (~~bend~~ based on web only)

$$\phi = \frac{d^2y}{dx^2} = \frac{\sigma_y}{E}\sqrt{\frac{\sigma_y t}{3(M_p - M)}} \qquad \text{(b)}$$

where σ_y = yield point of the material

M_p = plastic hinge moment

M = moment at the section

t = thickness of web

for portion BC ~~(straight line)~~

$$\phi = \frac{d^2y}{dx^2} = \frac{M - AZ}{BI} \qquad \text{(c)}$$

where A, B are constants of the material in the strain hardening range, and Z is the statical moment of the cross section

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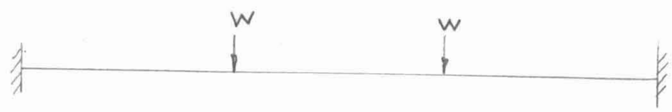
* See discussion of Weiskopf's paper, 205.H.

By using these three expressions and the continuity between these three regions one can solve an indeterminate structural problem. Take the built-in beam as an example.



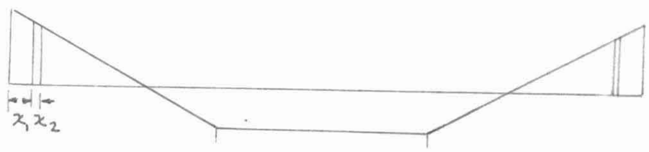


Fig e

what is M_A?

Suppose W brings the portion x shown above in the strain hardening range and the rest of the beam is in elastic range.

Since M_A and M_p in the M-Ø curve is known for given section, x_1 and x_2 can be computed in terms of M_F. If W is given, the only unknown is M_F in this case. By using the boundary conditions $\frac{dy}{dx} = 0$ at B and the continuity at section x_1 and x_2 the whole problem is solved. ~~The~~ An example is illustrated ~~on Sheet 9 in Appendix of the attached Weiskopf's report.~~ in the Appendix Equation (47) is of the form solved for M_F.

$$M_F^2 + K_1 M_F + K_2 = 0$$

in which K_1 and K_2 are known constants.

If the symmetry of loading is distrubed then the two end moments are both unknown. Though the same procedure can be applied to the solution, it will arrive to two simultaneous equations of second degree. Only a cut and try method can be used to solve the problem. The whole analysis will involve large amount of computing work.

Suggest some other letter to avoid confusion with B.M. $\frac{M-Az}{EI}$

Further ~~information~~ simplifications can be obtained by assuming the M-Ø relation as indicated in Fig. (d) with dotted lines O A' and A'B, which eliminates the plastic range in Weiskopf's equations. The beam is assumed only to have elastic (oA') pure plastic (A'-B) and strain hardening (B-C) regions with two sets of equations.

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

$$\frac{d^2y}{dx^2} = \frac{M - AZ}{BI}$$

These equations can be very easily integrated for most of the moment distributions". A large amount of computing work for solving the problem is being simplified. Since the plastic region as defined by Weiskopf is usually a very short portion in "I" beams the analysis with this further assumption is expected to be very close to Weiskopf's analysis.

Methods of deformation and strength for structure in plastic range discussed above can be summarized as follows:

Method b. is all new and has not been discussed above.

a. Deflections of structures can be solved approximately by the simple plastic theory assumptions when the deflection of the structure is small. Plastic hinges moments are put at sections which are yielded and deflections can be solved as in elastic structures. Strain hardening are not considered. It is impractical to apply it to the statically determinate structure in plastic range for deformations of those structures

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* See example worked in Appendix C.

usually very large when their initial yield strength is exceeded. In statically indeterminate structures, the design ultimate load is usually divided by factor of safety to obtain the working load. Under working load the structure may be within elastic range or partially in plastic range. If the structure is only partially in plastic range the deflections are ~~xxxxx~~ expected to be small and this approximate method is very likely to give satisfactory results.

b. Method of numerical integration: Theoretical M-ϕ curves with consideration of the strain harden effect can be computed when the geometric dimensions of the bending member and the stress-strain diagram of the material are known. For determinate structures deflections can be easily integrated numerically. In statically indeterminate structures solutions can be approached by successive relaxation. Take the built-in beam under concentrated load as an example:

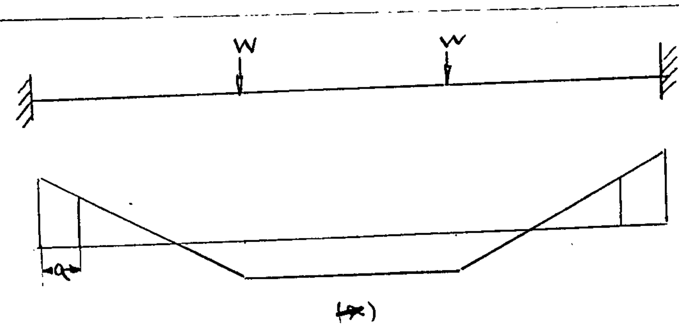

Fig f.

Consider the beam as elastic compute the moment distribution of the beam as shown in (f). At a distance "a" from either end may have moments higher than M_y. Find the conjugate beam with corresponding ∅ as the load by using ~~xx~~ of the computed M-∅ diagram.

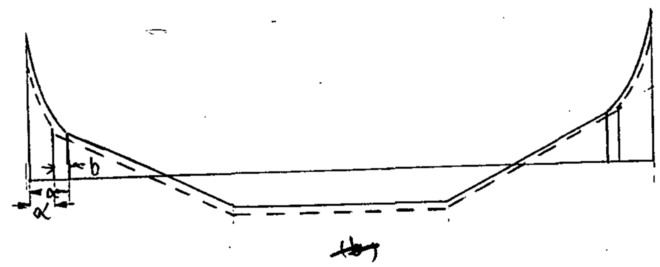

Fig h

The shear at both ends of this conjugate beam is of course not zero. But the slope of the loaded beam at both ends are zero. Shift the axis by numerical integration to make the integrated shear of the conjugate beam to be zero at both ends as shown in (h) with dotted lines, i.e., consider the loaded beam elastic again and apply moment at both ends to make the slope at both ends zero. The moment on portions b in Fig. (h) now become under M_y again. Construct a new conjugate beam with the computed M-∅ curve consider only the portions "a" near each end in plastic range as Fig. i).

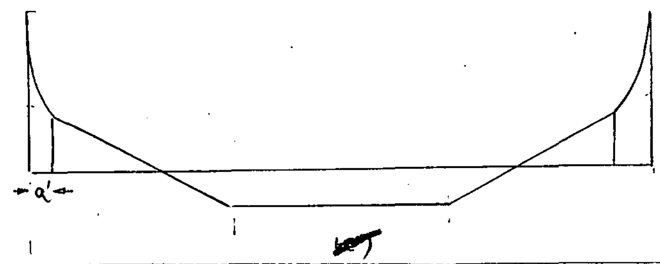

Fig i

Balance the conjugate beam again as above and repeat the process until satisfactory results are obtained.

This method gives a theoretically very logical solution. The only trouble is too much complicated computation especially the degree of redundance of the indeterminate structure increases.

c. Weiskopf's method of using simplified M-ϕ relations in plastic and strain hardening ranges for "I" beams: This method as discussed previously would be simpler than method 1(b) but still involves too much computing work in practical design.

d. The method of using approximate M-ϕ curve as shown in Fig. in dotted lines suggested by the author is actually the combination of method (a) and (c). The amount of calculation work is also between method (a) and (c). It seems reasonable results can be expected.

The above methods are compared in applying to the case of a cantilever beam of 8WF40 section with the span of 84" in Fig. 39. The experimental curves from test B2, B4, & B5 also presented in ~~Fig.~~ 39B, C, D for comparison. In Fig. 39 the deflections calculated by the method (a) neglecting the strain hardening give a discrepency to the average experimental* curve when the deflection becomes high. The curve computed by method (b), numerical integration, has a big discrepency with the experimental curve in the early plastic range on account of factors like stress concentration and residual stress. The curve computed by the same method but use the experimental M-ϕ curve from control test gives the best check with average experimental deflection curve in the early portion. The use of

Note: not shown on Fig 39

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* Data from six over hang cantilever beams of continuous beams 2, 4, 5.

M-ϕ curve observed in control beam test would eliminate some factors like rolling residual stress but not the factors like weld residual stresses and stress concentration. Therefore still there is a discrepency between these two curves in the early plastic range.

The curve computed by method (c) and (d) gives very close check with the curve computed by method (b). Among them method (d) is the simplest. If the M_p value in method (d) is slightly reduced by a certain abritrary criterion, say

$$\text{put } M'_p = \frac{M_p + M_y}{2}$$

and use M'_p instead of M_p the result can be made very close to check the experimental results. Experimental results of six cantilever beams in Fig. 39B, C, D actually shows a bigger scattering than the discrepancies among the curves computed by different theories.

For the case of indeterminate structures deflections at the central section of continuous beam 3 and beams 2, 4, 5 are computed by method (b) and (d) for comparison. The average experimental curve shown a lower strength than the cantilever beam is because of shear failure in the web which will be discussed in the next section.

Observed deflection in the central section for all the beams are plotted in Fig. 40A, B, C, D

The support detail of B5* is different from the rest of the beams which is one of the reasons that B5 shows a higher value of strength.

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*See Fig. 1

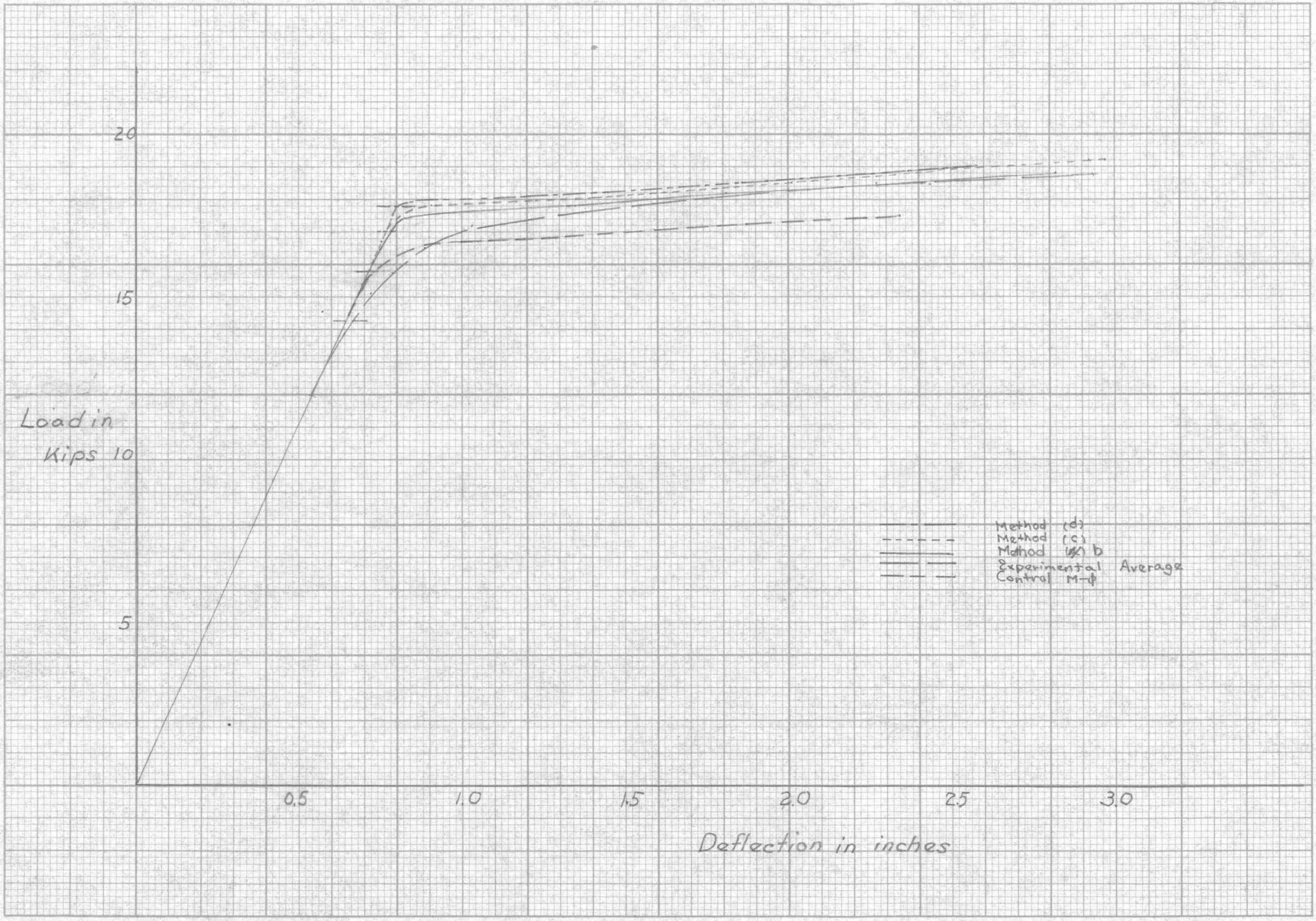

20
15
Load in
Kips 10
5
0.5
1.0
1.5
2.0
2.5
3.0
Deflection in inches
Method (d)
Method (c)
Method b
Experimental Average
Control M-φ

FIG 39

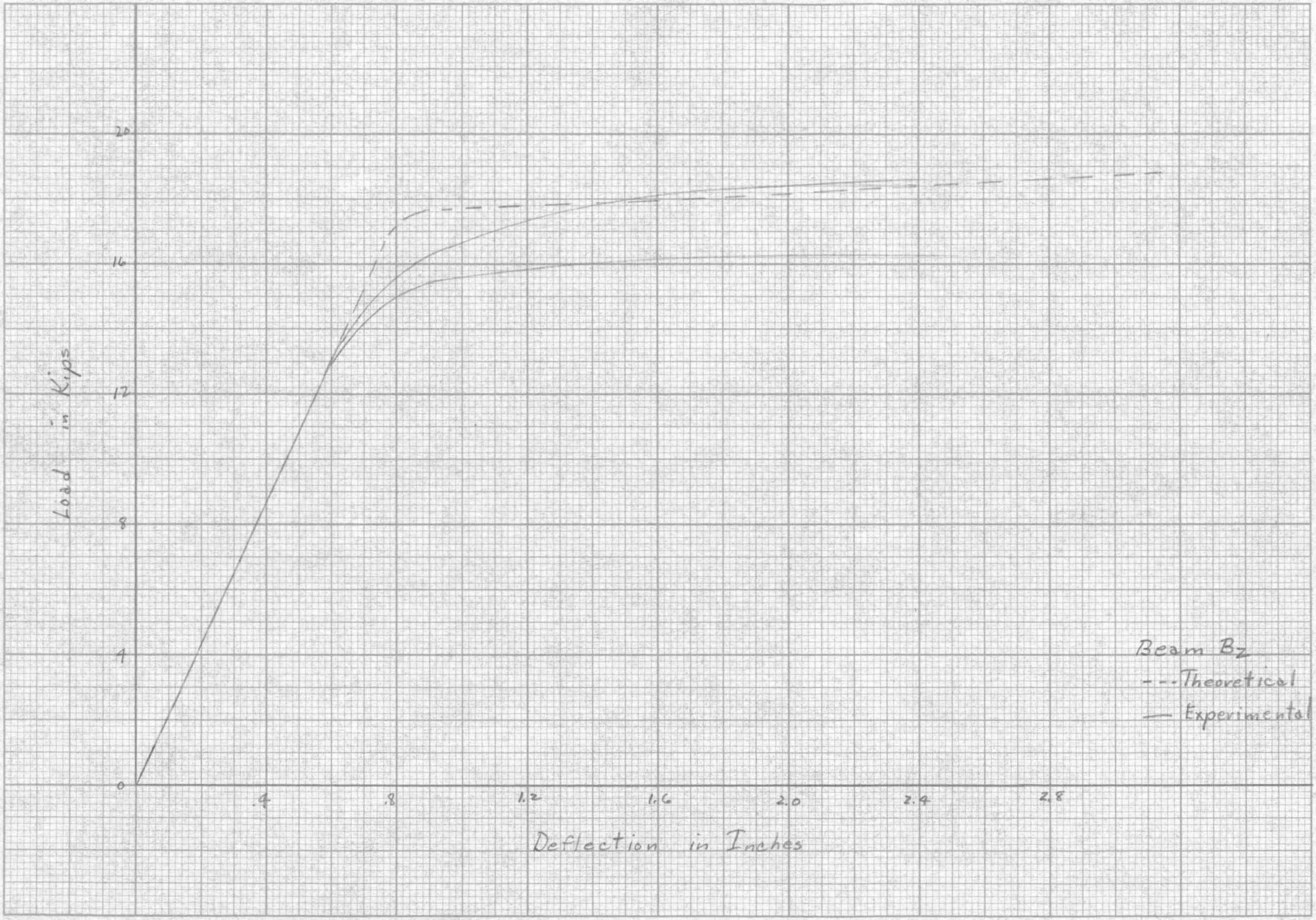

20
16
12
8
4
0
Load in Kips
.4
.8
1.2
1.6
2.0
2.4
2.8
Deflection in Inches
Beam B_2
--- Theoretical
— Experimental

FIG 39B

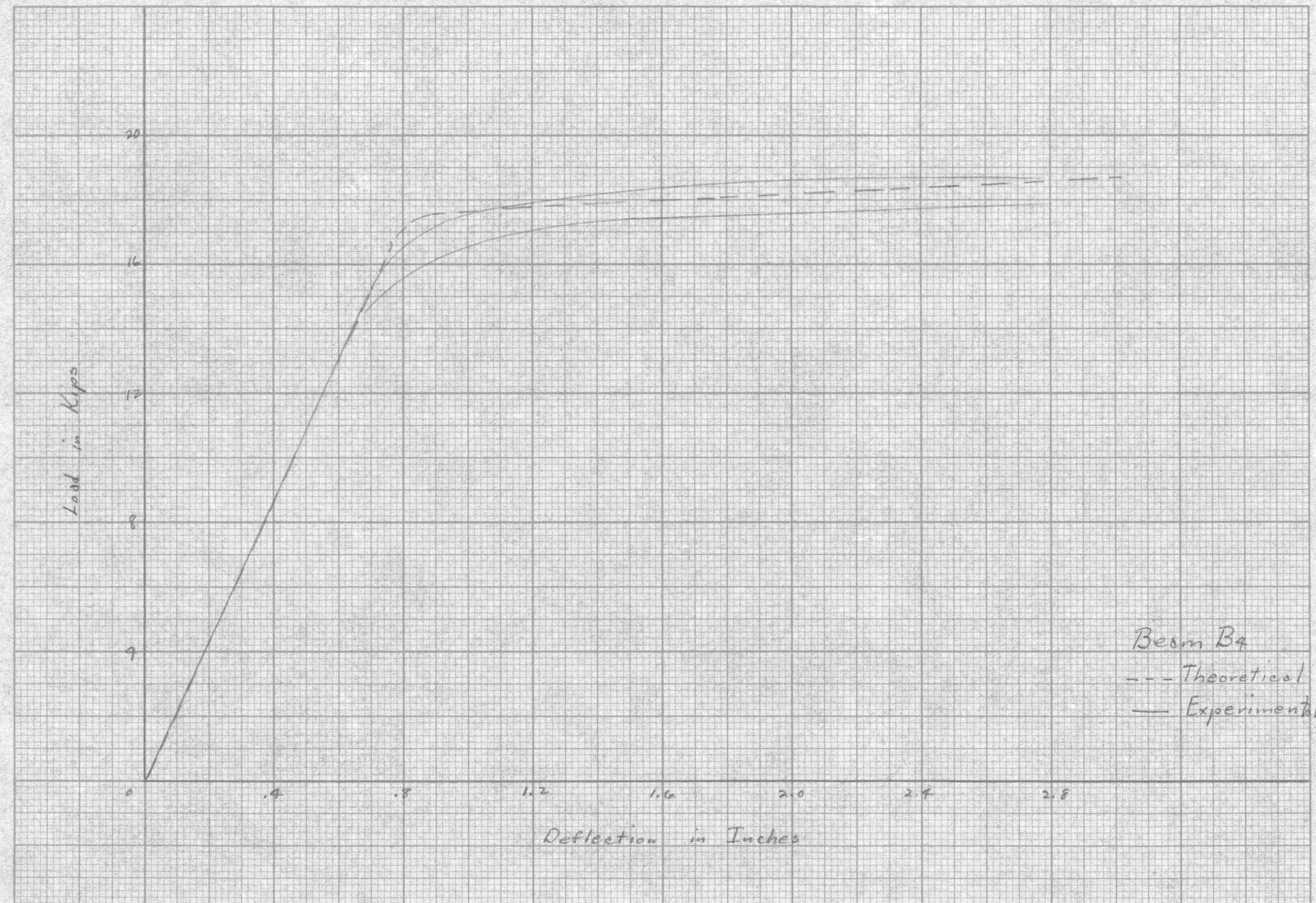

Load in Kips
20
16
12
8
4
0
.4
.8
1.2
1.6
2.0
2.4
2.8
Deflection in Inches
Beam B4
Theoretical
Experimental

Fig 39C

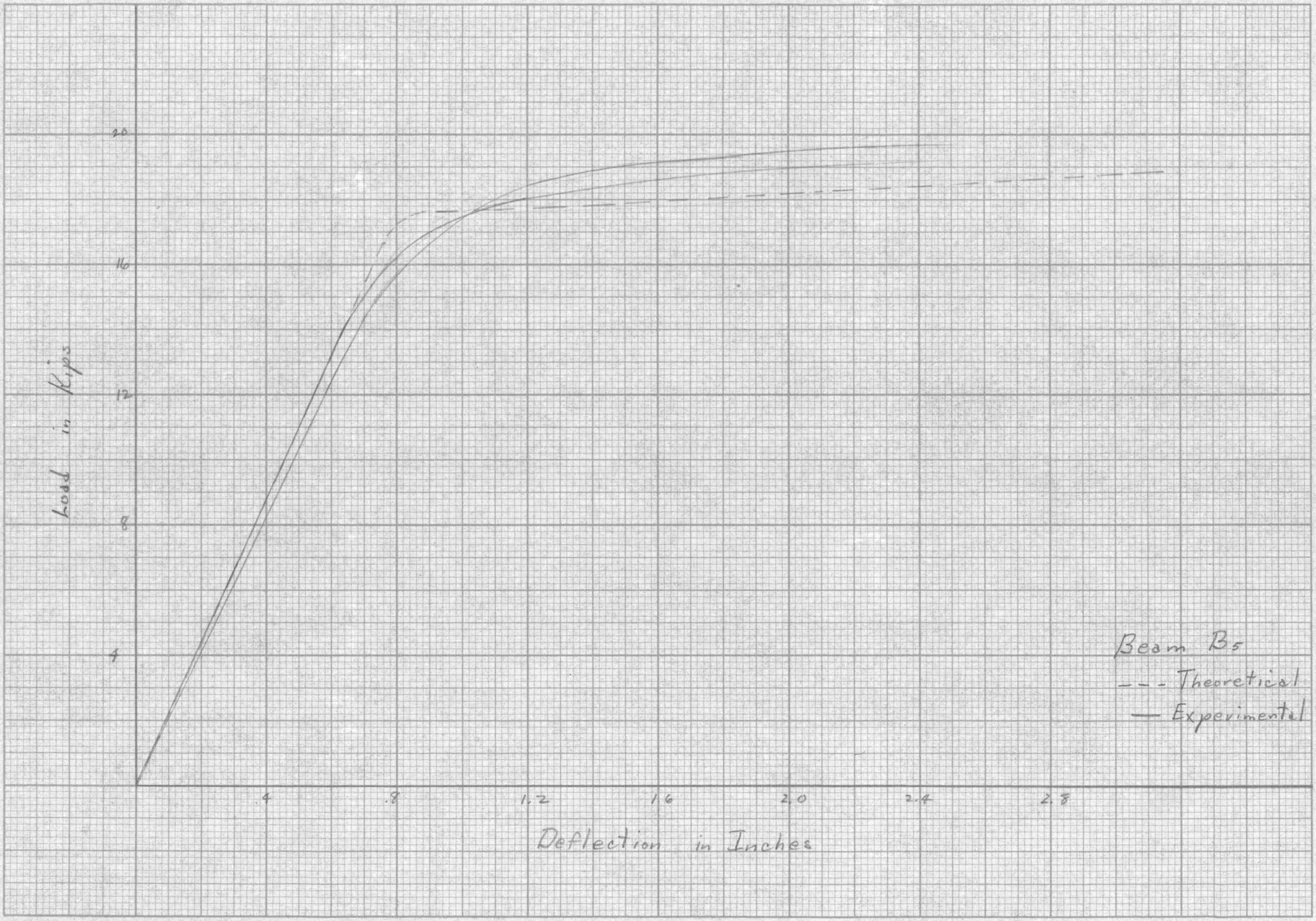

Load in Kips
20
16
12
8
4
.4
.8
1.2
1.6
2.0
2.4
2.8
Deflection in Inches
Beam B5
--- Theoretical
— Experimental

Fig 39D

NO. 340R-20 DIETZGEN GRAPH PAPER
20 X 20 PER INCH

EUGENE DIETZGEN CO.
MADE IN U.S.A.

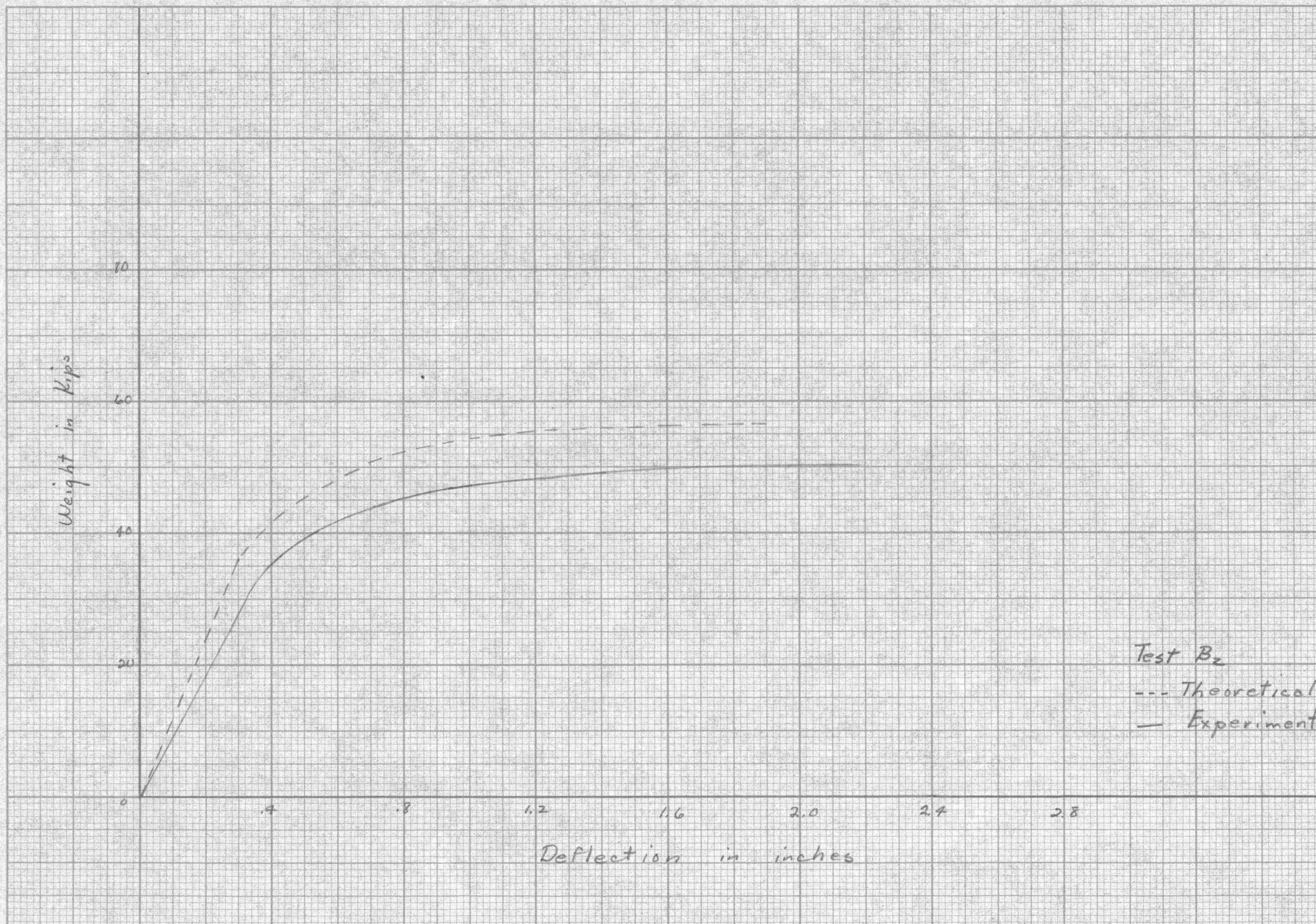


Fig 40A

NO. 340R-20 DIETZGEN GRAPH PAPER
20 X 20 PER INCH

EUGENE DIETZGEN CO.
MADE IN U. S. A.

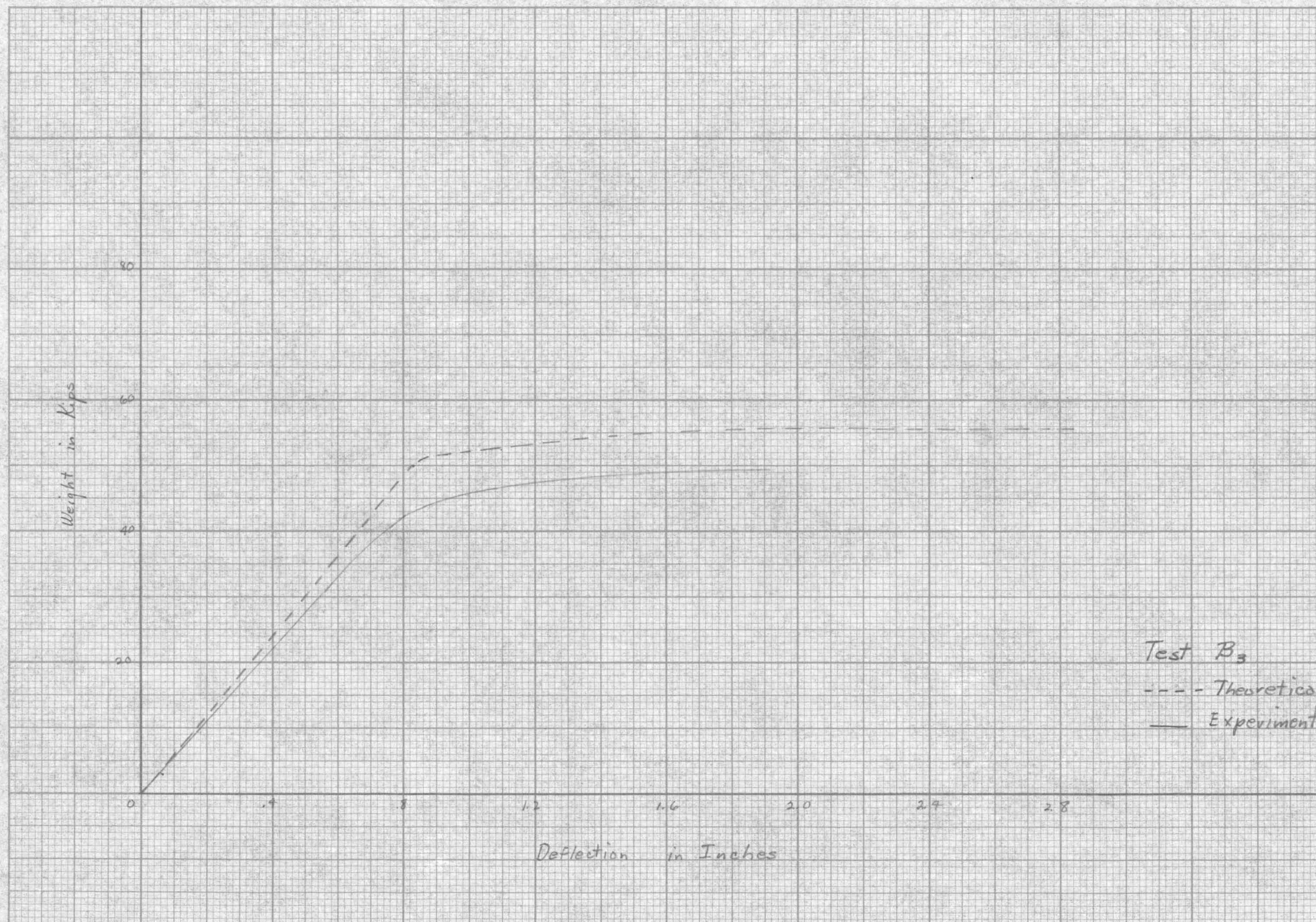


Fig 40 B

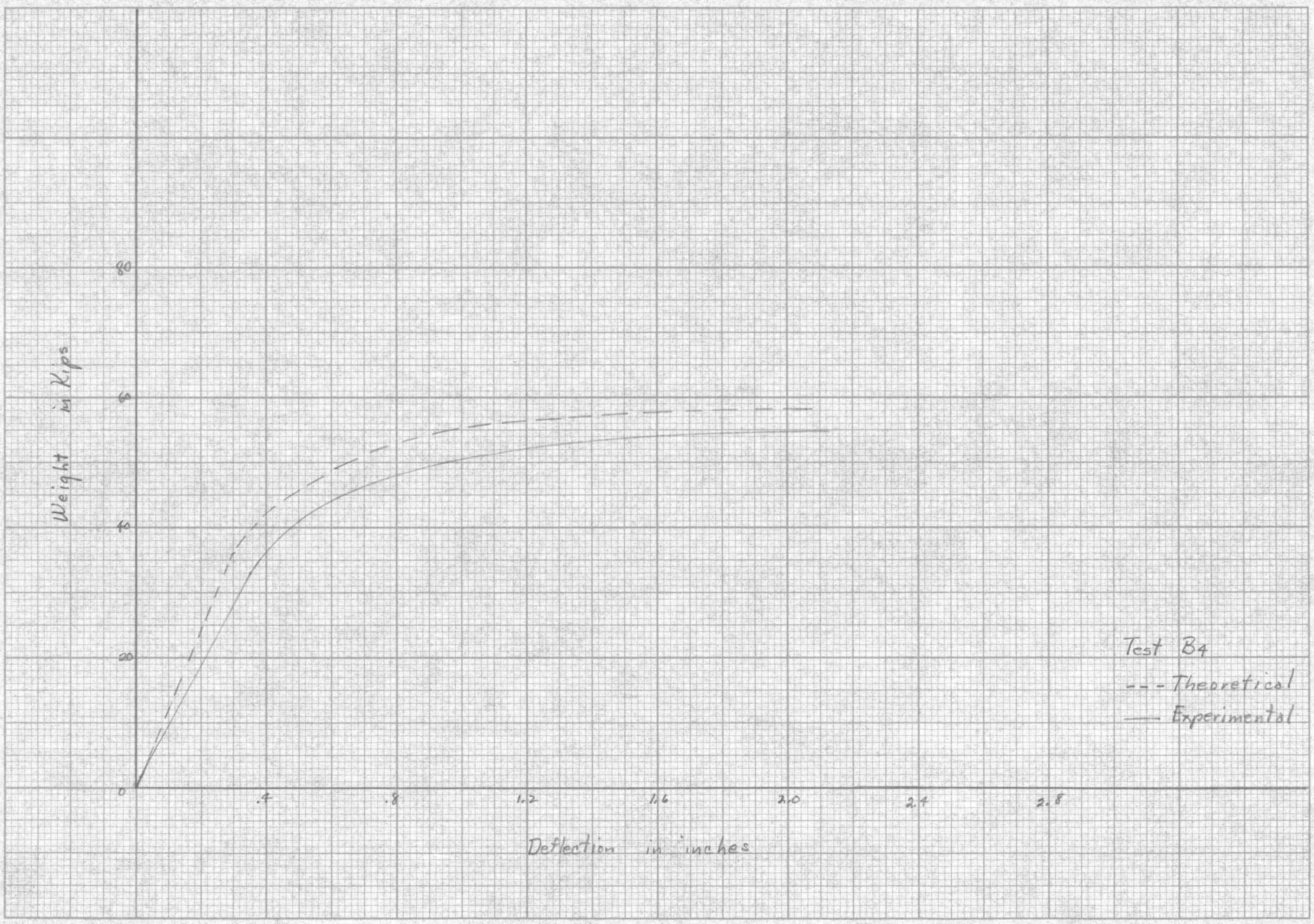
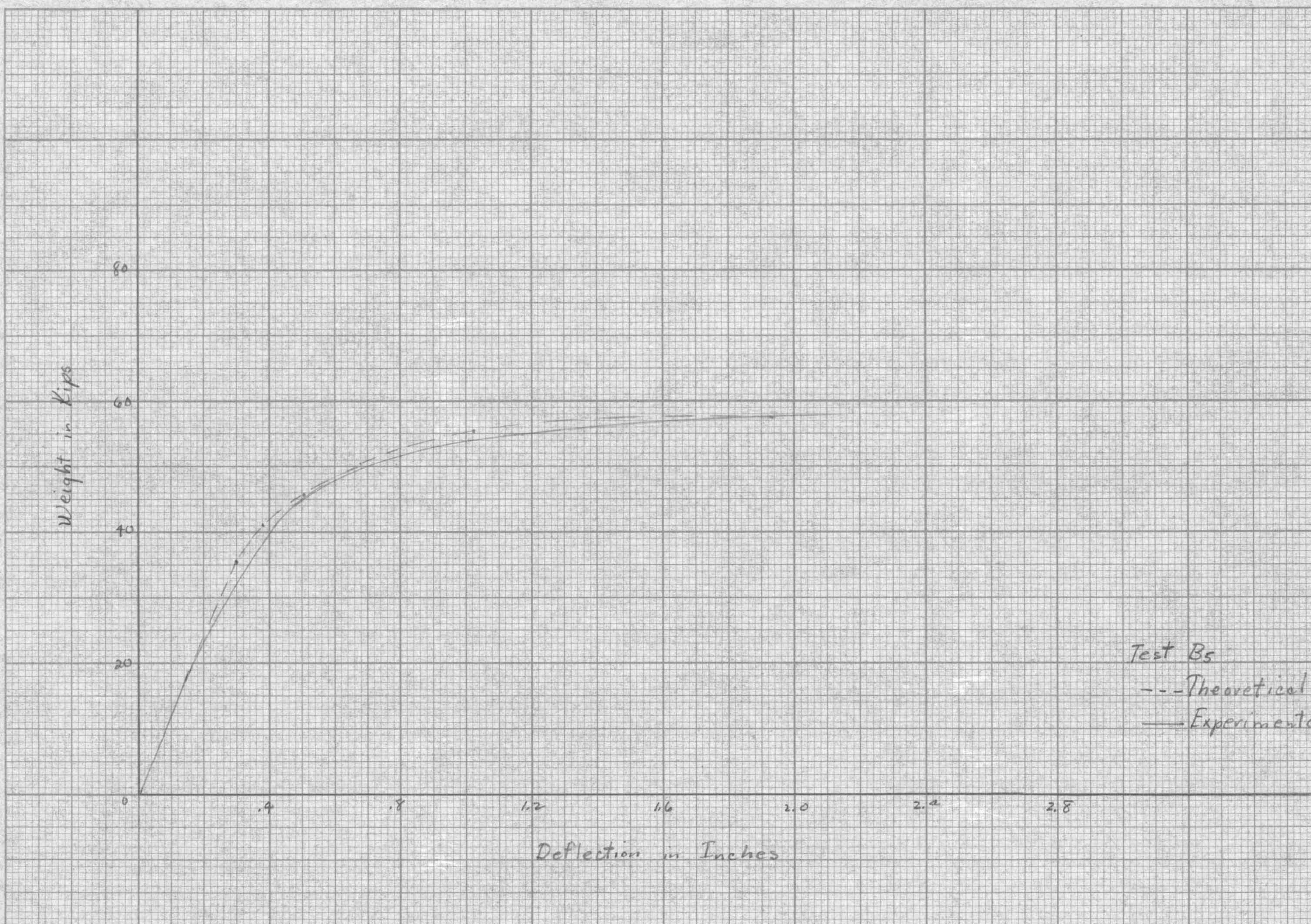

Weight in Kips
80
60
40
20
0
.4
.8
1.2
1.6
2.0
2.4
2.8
Deflection in inches
Test B4
- - - Theoretical
—— Experimental

FIG 40c


Weight in Kips
80
60
40
20
0
.4
.8
1.2
1.6
2.0
2.4
2.8
Deflection in Inches
Test B5
- - - Theoretical
—— Experimental

(Fig 40)